Generalized Plane Waves and Waveguide Modes in a Moving Isotropic Medium

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The differential equation for $\tilde{E}(z)$ [see Eq. (15)] now becomes

$$(\partial^2\tilde{E}/\partial z^2)+j\{[(\omega+\omega_c)/v_z]-\beta_0\}(\partial\tilde{E}/\partial z)-\alpha_0^2\tilde{E}=0, \tag{A8}$$

where α_0^2 is given by Eq. (23), with the appropriate β_0 [Eq. (A6)].

The final differential equation for the normalized field is exactly the same as for the synchronous case, [Eq. (26)], except that now

$$\delta=\frac{1}{2}\left[\frac{\beta_0-(\omega+\omega_c)/v_0}{\alpha_0}\right] \tag{A9}$$

and

$$Q=(\beta_0/\alpha_0)[(1+\omega_c/\omega)/8V_0I_0]P_{\mathrm{in}}. \tag{A10}$$

From Eq. (A10), we see that with cyclotron wave interactions, the maximum output power and electronic efficiency are reduced by a factor of $(1+\omega_c/\omega)^{-1}$ compared to the synchronous wave case. Physically, this reduction arises because a larger fraction of the circuit power goes into transverse kinetic energy of the electrons in the cyclotron wave case.

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Generalized Plane Waves and Waveguide Modes in a Moving Isotropic Medium

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The Lorentz transformation of plane-wave-like solutions and general waveguide modes is analyzed. A propagation and attenuation tensor is introduced. General Doppler equations and invariant phase quantities are shown to be simple consequences of the formalism. The concept of dispersion is discussed and a covariant condition connected with this concept is given. Covariant wave equations and dispersion relations are derived in a simple manner. The dispersion relations are used to analyze some special waveguide solutions including cutoff phenomena.

I. INTRODUCTION

This paper is concerned with the Lorentz transformation of certain solutions of Maxwell–Minkowski's equations including space–time attenuated plane waves and waveguide modes, the medium being homogeneous, isotropic, conducting, dispersive, and in uniform motion.

Since we are dealing with a single medium we may be guided by the basic idea that "all" problems are solved in the rest system of the medium [even in the case of usual waveguides because the relative motion between waveguide wall and medium does not effect the boundary conditions (consult Ref. 1)]. The task left is, therefore, to transform and reinterpret known results. The authors feel that this aspect has not been emphasized strongly enough in recent papers on the subject.[2–8]

We choose 4-dimensional tensor formalism which seems to be much more suitable for the purpose than traditional 3-dimensional vector formalism.

By a tensor we understand a tensor defined on the Lorentz transformation group. Latin subscripts run from 1 to 4, Greek subscripts run from 1 to 3. The coordinate $x_4=ict$, where t is the time and c the speed of light in vacuum; therefore the metric tensor in 4 space is equal to the Kronecker symbol δ_{ij} (when cartesian spatial coordinates are used) and we do not distinguish between contravariant and covariant tensors. Repeated subscripts obey the summation convention, and commas in subscripts denote partial differentiation with respect to coordinates (or covariant differentiation since the metric tensor is independent of the coordinates).

II. PLANE-WAVE-LIKE SOLUTIONS

Consider the transformation matrix a_{mn} for a proper Lorentz transformation,[9] i.e., $x_n=a_{rn}x_r^*$, $x_n^*=a_{nr}x_r$.

Without loss of generality we choose

$$a_{mn}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\gamma&i\beta\gamma\\0&0&-i\beta\gamma&\gamma\end{pmatrix}, \tag{1}$$

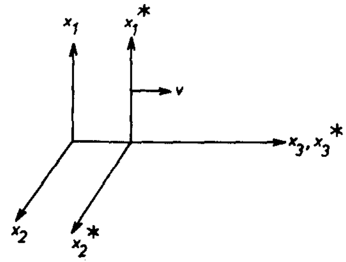

FIG. 1. Motion of K^* relative to K.

where $\gamma\equiv(1-\beta^2)^{-1/2}$, $\beta\equiv v/c$ and v is the velocity of K^* relative to K; the direction of v is given in Fig. 1.

Let k_n be a constant (i.e., independent of x_r) tensor of first order, and let f_{mn}, h_{mn} be antisymmetric constant tensors of second order. From these tensors we define the antisymmetric tensorfields

$$f_{mn}\exp(ik_rx_r) \tag{2a}$$

$$h_{mn}\exp(ik_rx_r). \tag{2b}$$

In Minkowski's theory the electromagnetic field in a moving medium can be described by means of two antisymmetric tensors of second order[1]:

$$F_{mn}=\begin{pmatrix} 0 & cB_3 & -cB_2 & -iE_1 \\ -cB_3 & 0 & cB_1 & -iE_2 \\ cB_2 & -cB_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix},$$

$$\tilde{F}_{mn}=\begin{pmatrix} 0 & -iE_3 & iE_2 & cB_1 \\ iE_3 & 0 & -iE_1 & cB_2 \\ -iE_2 & iE_1 & 0 & cB_3 \\ -cB_1 & -cB_2 & -cB_3 & 0 \end{pmatrix} \tag{3a}$$

$$H_{mn}=\begin{pmatrix} 0 & H_3 & -H_2 & -icD_1 \\ -H_3 & 0 & H_1 & -icD_2 \\ H_2 & -H_1 & 0 & -icD_3 \\ icD_1 & icD_2 & icD_3 & 0 \end{pmatrix},$$

$$\tilde{H}_{mn}=\begin{pmatrix} 0 & -icD_3 & icD_2 & H_1 \\ icD_3 & 0 & -icD_1 & H_2 \\ -icD_2 & icD_1 & 0 & H_3 \\ -H_1 & -H_2 & -H_3 & 0 \end{pmatrix}, \tag{3b}$$

where the dual tensors are given by $\tilde{F}_{mn}\equiv(1/2!)\epsilon_{mnrs}F_{rs}$, for example.[10]

It is well known that for a homogeneous, isotropic, dispersive, and conducting medium without sources there exist solutions of Maxwells equations which in the rest system K' of the medium are of the form as given by (2).

Because we are dealing with tensors these solutions are in any system of inertia given by

$$F_{mn}=f_{mn}\exp(ik_rx_r) \tag{4a}$$

$$H_{mn}=h_{mn}\exp(ik_rx_r) \tag{4b}$$

which we call PWL solutions. k_r are complex numbers.

III. SOME TRANSFORMATION PROPERTIES

Any tensor can be split into a sum of two tensors. We choose two tensors λ_n and α_n so that

$$k_n=\lambda_n+i\alpha_n \tag{5}$$

and define $\lambda_\nu\equiv\mathrm{Re}(k_\nu)$, $\lambda_4\equiv i\,\mathrm{Im}(k_4)$, $\alpha_\nu\equiv\mathrm{Im}(k_\nu)$, $\alpha_4\equiv-i\,\mathrm{Re}(k_4)$. This definition is consistent (i.e., independent of the inertial system) which is readily seen from (1).

We call λ_n propagation tensor and α_n attenuation tensor and the quantity

$$\omega\equiv-ic\lambda_4 \tag{6}$$

is the frequency.

Depending on the values of λ_n and α_n (4) can be interpreted as a (non)homogeneous plane wave, (non)attenuated in some/all coordinates or (4) cannot be interpreted as a travelling wave at all. The interpretation depends on the inertial system as discussed in Ref. 8 and is in principle settled by the fact that λ_n and α_n are tensors.

The relativistic Doppler equations[8] are given by λ_n, α_n and the tensor transformation law:

$$\lambda_n=a_{rn}\lambda_r^*,\qquad \lambda_n^*=a_{nr}\lambda_r \tag{7a}$$

$$\alpha_n=a_{rn}\alpha_r^*,\qquad \alpha_n^*=a_{nr}\alpha_r. \tag{7b}$$

If λ_ν is proportional with α_ν (i.e., the wave is homogeneous) in one system of inertia this is obviously in general not the case in another intertial system. (Of course, if λ_n is proportional with α_n this proportionality is independent of the inertial frame.)

The tensor product

$$\Phi\equiv k_nx_n \tag{8}$$

is invariant, and substituting (5) into (8) we get the invariants

$$\mathrm{Re}(\Phi)=\lambda_nx_n \tag{9a}$$

$$\mathrm{Im}(\Phi)=\alpha_nx_n. \tag{9b}$$

If $\lambda_\nu\lambda_\nu\neq0(\alpha_\nu\alpha_\nu\neq0)$, (9) can be used to define a propagation (attenuation) phase plan and velocity, e.g., $v_p=-ic\lambda_4/(\lambda_\nu\lambda_\nu)^{1/2}=\omega/(\lambda_\nu\lambda_\nu)^{1/2}$.

Similarily with (5) the tensors f_{mn}, h_{mn} can be split into two tensors:

$$f_{mn}=a_{mn}+ib_{mn} \tag{10a}$$

$$h_{mn}=d_{mn}+ie_{mn}, \tag{10b}$$

where the quantities on the right sides are antisymmetric tensors given by $a_{\mu\nu}\equiv\mathrm{Re}(f_{\mu\nu})$, $a_{m4}\equiv i\,\mathrm{Im}(f_{m4})$, $b_{\mu\nu}\equiv\mathrm{Im}(f_{\mu\nu})$, $b_{m4}\equiv -i\,\mathrm{Re}(f_{m4})$. ($d_{mn}$, e_{mn} are defined in quite the same way by means of h_{mn}.) Again this definition is seen to be consistent.

Phase angles $\phi_{(mn)}$ may be defined by

$$\tan\phi_{(mn)}\equiv b_{mn}/a_{mn} \tag{11a}$$

$$\tan\psi_{(mn)}\equiv e_{mn}/d_{mn}. \qquad \text{(no summation)} \tag{11b}$$

It is noticed that phase angles (or differences between two of them) in general are not Lorentz invariants as pointed out in Ref. 8 but they are readily found by means of (11) in any system of inertia.

A PWL solution is easily transformed from one system of inertia to another. By inspection of the transformation matrix (1) we conclude that an element of any tensor (of arbitrary order) is uneffected by the Lorentz transformation if none of the suffix values exceed the integer 2, e.g., $k_2{}^*=k_2$, $h_{12}{}^*=h_{12}$.

Furthermore, if a tensor T_{mn} is antisymmetric then $T_{34}{}^*=T_{34}$ [and therefore also $T_{43}{}^*=T_{43}$ because (anti)-symmetry is an invariant property], for

$$T_{34}{}^*=a_{3r}a_{4s}T_{rs}=a_{33}a_{4s}T_{3s}+a_{34}a_{4s}T_{4s}$$
$$=a_{33}a_{44}T_{34}+a_{34}a_{43}T_{43}=T_{34}(a_{33}a_{44}-a_{34}a_{43})=T_{34}$$

because $\det(a_{rs})=1$.

Applying these statements to the field tensors (3) we see that all field components with x_3 direction are Lorentz invariant (as it should be).

Finally it is noticed that due to the zeroes in (1) the amount of calculation work is considerably reduced when a tensor is to be transformed.

IV. DISPERSION

The concept of dispersion is usually based on the assumption that the field can be decomposed into time-harmonic components in the rest system K' of the medium. Such a component can be written as

$$g'(x_\nu')\exp(il_4'x_4'), \tag{12}$$

where l_4' is a constant and g' is a function independent of the time coordinate x_4'. The frequency is given by $\omega'\equiv -icl_4'$ [cf. (6)]. Furthermore ω' has to be real, i.e.,

$$\mathrm{Im}(\omega')=\mathrm{Re}(l_4')=0. \tag{13}$$

PWL solutions as given by (4) are obviously time harmonic in K' if

$$\alpha_4'=0. \tag{14}$$

This equation can be translated into covariant tensor language. To do this we make use of the velocity tensor U_n of an inertial system K with respect to the rest system K' which in K' is given by[9]

$$U_n'=(0,0,0,ic). \tag{15}$$

Since $\alpha_4'=\alpha_r'U_r'/ic=\alpha_rU_r/ic$, (14) may be written as

$$\alpha_rU_r=0. \tag{16}$$

By means of (6) and (15) we derive $\omega'=-\lambda_4'U_4'=-\lambda_r'U_r'$, i.e.,

$$\omega'=-\lambda_rU_r. \tag{17}$$

If the PWL solution is time harmonic in K' we get from (5), (16), and (17) that

$$\mathrm{Im}(k_rU_r)=0, \tag{18}$$

which is equivalent with the condition (14) or (16), and that the frequency in K' is given by

$$\omega'=-k_rU_r. \tag{19}$$

In Minkowski's theory the constitutive parameters are invariant quantities, i.e., in the case of a dispersive medium $\epsilon'(\omega')=\epsilon(\omega')$, say. Using (17) or (19) this function may be written with covariant argument as $\epsilon(-\lambda_rU_r)$ or $\epsilon(-k_rU_r)$.

Though a PWL solution remains a PWL solution under a Lorentz transformation the condition $\alpha_4=0$ (time harmonic in K) alone does not imply $\alpha_4{}^*=0$ in another inertial system. In fact only if $\alpha_3=\alpha_4=0$ (no attenuation in the x_3 and x_4 coordinate) then $\alpha_3{}^*=\alpha_4{}^*=0$ as seen by inspection of (1).

Consider a PWL "solution" with $\alpha_4=0$, $\alpha_3\neq 0$ in K different from K', i.e., $\alpha_4'\neq 0$. If we are working with a dispersion model as described above we have to show that the "solution" can be decomposed into time-harmonic components in K'. This leads to convergency problems. Obviously the "solution" exists if we work with a dispersion model as suggested in Ref. 8, where ϵ (for example) is a function of the complex variable $-ick_4'=-ic(\lambda_4'+i\alpha_4')=-k_rU_r$, because condition (14), (16), (18) is no longer necessary, but the choice of a dispersion model is not only a question of mathematical convenience.

V. THE WAVE EQUATION

Not every set of tensorfields (2) represents a PWL solution; the Maxwell–Minkowski's equations have to be satisfied.

In the rest system K' of the medium the well-known condition ("dispersion relation")

$$k_\nu'k_\nu'+n^2k_4'^2-\sigma\mu ck_4'=0 \tag{20}$$

is obtained by substituting (2) into the wave equation

$$\left(\nabla'^2+n^2\frac{\partial^2}{\partial x_4'^2}-ic\sigma\mu\frac{\partial}{\partial x_4'}\right)\begin{Bmatrix}F_{mn}'\\ H_{mn}'\end{Bmatrix}=0. \tag{21}$$

In order to get covariant formulations of (20) and (21) one may derive the tensor wave equation in K from the tensor formulation of Maxwell–Minkowski's equations. Another way is to translate (21) directly into tensor language which will be done here.

From (21) we get

$$T_{,mm}'+(n^2-1)T_{,44}'-ic\sigma\mu T_{,4}'=0, \tag{22}$$

which by means of (15) can be written as

$$T_{,mm}-\kappa T_{,mn}U_mU_n-\sigma\mu T_{,m}U_m=0, \tag{23}$$

where T stands for a field tensor and

$$\kappa \equiv (n^2-1)/c^2. \tag{24}$$

We have omitted the primes in (23) because it is a tensor equation.

In a similar way or by inserting (4) into (23) we get the condition

$$\Lambda = 0, \tag{25}$$

where

$$\Lambda \equiv k_m k_m - \kappa (k_m U_m)^2 + i\sigma\mu (k_m U_m). \tag{26}$$

Equation (25) ensures that (2) is a solution of (23).

From (5) and (26) we finally derive the real invariants

$$\mathrm{Re}(\Lambda) = \lambda_m\lambda_m - \alpha_m\alpha_m - \kappa[(\lambda_m U_m)^2 - (\alpha_m U_m)^2] - \sigma\mu\alpha_m U_m \tag{27a}$$

$$\mathrm{Im}(\Lambda) = 2\lambda_m\alpha_m - 2\kappa\lambda_m U_m \alpha_n U_n + \sigma\mu\lambda_m U_m. \tag{27b}$$

VI. GENERALIZED PLANE-WAVE-LIKE SOLUTIONS

PWL solutions as given by (4) are not very useful if we are concerned with waveguides containing a moving medium. To this purpose we will look at field solutions which can be written in the form

$$F_{mn} = f_{mn}(x_1, x_2) \exp(ik_r x_r) \tag{28a}$$

$$H_{mn} = h_{mn}(x_1, x_2) \exp(ik_r x_r), \qquad k_1 = k_2 = 0, \tag{28b}$$

where f_{mn}, h_{mn} are tensorfields independent of x_3, x_4, and k_r is a constant tensor with zero first and second coordinate. From the remarks made in Secs. II and III it becomes evident that this definition is consistent.

The results from Secs. III and IV may be extended to solutions as given by (28), in particular the tensor k_r can be split into two parts as defined by (5) (with $\lambda_1 = \lambda_2 = \alpha_1 = \alpha_2 = 0$).

Condition (25) in Sec. V must be generalized. Substituting (28) into (23) leads to the 2-dimensional wave equation

$$(\nabla_t^2 - \Lambda)\begin{cases} f_{mn}(x_1, x_2) \\ h_{mn}(x_1, x_2) \end{cases} = 0, \tag{29}$$

where the invariant Λ is still given by (26) and $\nabla_t^2 \equiv (\partial^2/\partial x_1^2) + (\partial^2/\partial x_2^2)$ is covariant under a Lorentz transformation given by (1).

VII. WAVEGUIDES

Consider a waveguide parallel to the x_3 direction with constant (i.e., independent of x_3) cross section and containing a uniformly moving, isotropic medium.

As in Sec. II our point of departure is the rest system K' of the medium where "all" problems are solved. We do this because the 2-dimensional wave equation (29) is valid in every systèm of inertia and the boundary conditions are not effected by the motion of the waveguide walls.[1]

In K' we know that solutions as given by (28) exist. These solutions are usually classified as TE and TM modes. Since $k_r' x_r' = k_r x_r$ all these modes can readily be transformed to another inertial system as outlined in Sec. III. Some special transformation properties of the solutions under consideration may be worthwhile mentioning here:

Since $E_3' = E_3 (D_3' = D_3)$, $H_3' = H_3 (B_3' = B_3)$ the concept of TE and TM waves is seen to be invariant.

Consider TE modes, i.e., $E_3 = 0$ or $\tilde{F}_{12} = 0$ [cf. (3a)]. It follows from the Maxwell equations[1,9]

$$\tilde{F}_{mn,n} = 0 \tag{30}$$

in connection with (28) that

$$\tilde{f}_{14} = -(k_3/k_4)\tilde{f}_{13} \tag{31a}$$

$$\tilde{f}_{24} = -(k_3/k_4)\tilde{f}_{23} \tag{31b}$$

in any inertial system.

From (31) and the transformation matrix (1) we easily obtain

$$\tilde{f}_{13} = \alpha' \tilde{f}_{13}' \tag{32a}$$

$$\tilde{f}_{23} = \alpha' \tilde{f}_{23}', \tag{32b}$$

where

$$\alpha' \equiv [a_{33} - a_{43}(k_3'/k_4')]. \tag{33}$$

Furthermore (1) shows that

$$k_4 = \alpha' k_4'. \tag{34}$$

It is well known that in K' all field components can be expressed by $\tilde{f}_{34}'$ (corresponding to B_3'). As to E_1', E_2' we have

$$i\Lambda \tilde{f}_{23}' = k_4' \tilde{f}_{34,2}' \tag{35a}$$

$$i\Lambda \tilde{f}_{13}' = k_4' \tilde{f}_{34,1}'. \tag{35b}$$

The transformation of (35) is now trivial. Obviously $\tilde{f}_{34}'(x_1', x_2') = \tilde{f}_{34}(x_1, x_2)$ for all $x_1' = x_1$, $x_2' = x_2$ (cf. Sec. VI) and by means of (32) and (34) we see that (35) are covariant equations (i.e., the primes can be omitted; this also holds for the equations expressing E_1', E_2' in terms of B_3' because $k_r x_r$ is invariant).

From (31) we deduce that the equations expressing B_1', B_2' in terms of B_3' are covariant.

As to TM modes we finally conclude by means of symmetry considerations that the equations expressing H_1', H_2', D_1', D_2' in terms of D_3' are covariant if $\sigma = 0$.

VIII. SPECIAL WAVEGUIDE SOLUTIONS, CUTOFF

Consider a waveguide mode specified by the condition

$$\Lambda = -\Gamma^2, \tag{36}$$

where the real mode number Γ^2 is determined by the covariant, 2-dimensional wave equation (29) in connection with the configuration of the waveguide cross section.

Λ is given by (26) or (27), i.e., (36) represents two relations between the 4 nonvanishing components λ_3,

λ_4, α_3, and α_4 of the propagation and attenuation tensor, necessary and sufficient conditions for the existence of a solution.

Solutions belonging to a given mode (i.e., Γ^2) including cutoff phenomena can readily be analyzed by means of these relations. We will do this for a nondispersive, lossless medium in two cases, the solutions being time harmonic in the rest system K' or in another inertial system K.

Case I: $\alpha_4'=0$, $\sigma=0$.

The transformation formula of the attenuation tensor is reduced to [cf. (1) and Fig. 1, K' coincident with K^*]

$$\alpha_4=i\beta\alpha_3 \tag{37a}$$

$$\alpha_3=\gamma\alpha_3'. \tag{37b}$$

Using (27), (6), and (15) condition (36) can be expressed by the well-known equations

$$\lambda_3'^2-\alpha_3'^2=n^2(\omega'/c)^2-\Gamma^2 \tag{38a}$$

$$\lambda_3'\alpha_3'=0, \tag{38b}$$

and there is no need for analyzing the covariant edition of (38).

Above cutoff, i.e., when $(\omega'/c)>(\Gamma/n)$ we obviously have $\alpha_3'=0$ and therefore $\alpha_3=\alpha_4=0$. The solutions are nonattenuated (harmonic) in the space and the time coordinate.

λ_3' is given by (38a) with $\alpha_3'=0$ and may be positive or negative (corresponding to a wave travelling in the positive or negative x_3' direction). Finally λ_3 and λ_4 are given by the tensor transformation law which can be written as

$$\lambda_3=\gamma[\lambda_3'+\beta(\omega'/c)] \tag{39a}$$

$$\omega/c=\gamma[(\omega'/c)+\beta\lambda_3']. \tag{39b}$$

From (39a) it is seen that if $\beta\lambda_3'<0$, then $\lambda_3=0$ is possible and the corresponding frequency in K' is given by $\omega'/c=\Gamma/(n^2-\beta^2)^{1/2}(>\Gamma/n$, the cutoff frequency in $K')$. The field is time harmonic but independent of x_3. This is a relativistic effect; the wave has been "run down" in such a way that a field component is in phase at all points in the x_3 direction. (Simultaneousness depends on the system of inertia!) One may call this effect cutoff, but the nature of this relativistic phenomenon is quite different from the well-known cutoff effect of a waveguide in the rest system K' of the medium.

$\omega=0$ when $\omega'c=|\beta|\Gamma/(n^2\beta^2-1)^{1/2}$ and $\beta\lambda_3'<0$ which is only possible when $(n\beta)^2>1$ (i.e., above the Cerenkov velocity as it should be).

Below cutoff in K', i.e., when $\omega'/c<\Gamma/n$ we have $\lambda_3'=0$ so that (39) reduces to

$$\lambda_3=\gamma\beta(\omega'/c) \tag{40a}$$

$$\omega/c=\gamma(\omega'/c). \tag{40b}$$

In this case α_3' is different from zero and given by (38a) implying that $\alpha_3\neq0$ and $\alpha_4\neq0$ [cf. (37)]. This means that the solutions represent space–time attenuated waves in K.

It is a relativistic effect that $\lambda_3\neq0$ (in spite of $\lambda_3'=0$) and that $\alpha_4\neq0$ (in spite of $\alpha_4'=0$). When the speed v of K relative to the medium becomes comparable with the speed of light in vacuum (i.e., $\beta^2\approx1$) then $|\lambda_3|\approx\omega/c$ and no cutoff effect is therefore observed in K.

Case II: $\alpha_4=0$, $\sigma=0$.

We readily derive

$$\alpha_4'=-i\beta\alpha_3' \tag{41a}$$

$$\alpha_3=(1/\gamma)\alpha_3' \tag{41b}$$

and

$$\lambda_3'^2-n^2(\omega'/c)^2-\alpha_4'^2[n^2-(1/\beta^2)]+\Gamma^2=0 \tag{42a}$$

$$\alpha_3'[\lambda_3'+n^2\beta(\omega'/c)]=0. \tag{42b}$$

From (42b) we see that $\alpha_3'=0$ or

$$\lambda_3'=-n^2\beta(\omega'/c). \tag{43}$$

Since $\alpha_3'=0$ implies $\alpha_3=\alpha_4=0$ in all systems K we already have treated this situation under case *I*.

Assuming that $\alpha_3'\neq0$ we strictly speaking leave our basic idea as mentioned in the introduction. When β^2 is comparable with 1 we are not "familiar" with the problem of how to excite a solution which is time harmonic in K (but not in K') whether the source is or is not at rest in K'! Nevertheless we may discuss the equations.

Inserting (43) into (42a) leads to the equation

$$-(\alpha_4')^2=n^2\beta^2\left[\frac{(\Gamma/n)^2}{1-n^2\beta^2}-\left(\frac{\omega'}{c}\right)^2\right]. \tag{44}$$

Since $-(\alpha_4')^2>0$ we get the conditions that $n^2\beta^2<1$ and $\omega'/c<(\Gamma/n)/(1-n^2\beta^2)^{1/2}$. Finally from (39) and (42) we obtain

$$\lambda_3=-\gamma\beta(n^2-1)(\omega'/c) \tag{45a}$$

$$\omega/c=\gamma(1-n^2\beta^2)(\omega'/c) \tag{45b}$$

which in turn leads to

$$\lambda_3=-\beta(n^2-1)/(1-n^2\beta^2)(\omega/c). \tag{46}$$

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